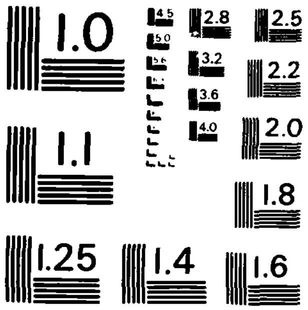
AD-A131 214   A MEASURE OF THE CONFORMITY OF A PARAMETER SET TO A   1/1
TREND: THE PARTIALLY..(U) IOWA UNIV IOWA CITY DEPT OF
STATISTICS AND ACTURIAL SCIENCE.. T ROBERTSON ET AL.

UNCLASSIFIED   MAY 83 N00014-80-C-0321   F/G 12/1   NL

END
DATE
FILMED
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS - 1963 - A

ADA131214

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO. AD-A131214
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) A MEASURE OF THE CONFORMITY OF A PARAMETER SET TO A TREND: THE PARTIALLY ORDERED CASE	5. TYPE OF REPORT & PERIOD COVERED Interim
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Dept. of Statistics, Univ. of Iowa, Iowa City, Iowa 52242; Dept. of Math. and Stat., University of Missouri, Rolla, MO 65401	8. CONTRACT OR GRANT NUMBER(s) N00014-80-C-0321, N00014-80-C-0322
9. PERFORMING ORGANIZATION NAME AND ADDRESS Dept. of Statistics, Univ. of Iowa, Iowa City, Iowa 52242; Dept. of Math. and Stat., University of Missouri, Rolla, MO 65401	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research, Department of the Navy, Arlington, VA 22217	12. REPORT DATE May 1983
	13. NUMBER OF PAGES 14
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

DTIC ELECTE AUG 10 1983 S D

18. SUPPLEMENTARY NOTES

D

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Order restricted inference, tests for and against a trend, isotonic inference, chi-bar-squared distributions, E-bar-squared distributions.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Inferences concerning order restrictions on a collection of parameters, $\theta_1, \theta_2, \ldots, \theta_k$, are considered with the order restrictions of the form, $\theta_i \le \theta_j$ for $i \preceq j$ where $\preceq$ is a partial order on $\{1,2,\ldots,k\}$. Clearly, some parameter sets conform more closely to these order restrictions than others. We are interested in measures of the degree of conformity. Some of the

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

DTIC FILE COPY

83 08 09 004

measures available in the literature for the totally ordered case are generalized to the partially ordered case and the theory developed is applied to several tests of order restricted hypotheses.

A MEASURE OF THE CONFORMITY OF A PARAMETER SET TO A TREND: THE PARTIALLY ORDERED CASE[1]

Tim Robertson
University of Iowa
Iowa City, Iowa 52242

F. T. Wright
University of Missouri-Rolla
Rolla, Missouri 65401

Abbreviated Title: Conformity to a Trend

ABSTRACT

Inferences concerning order restrictions on a collection of parameters, $\theta_1, \theta_2, \ldots, \theta_k$, are considered with the order restrictions of the form, $\theta_i \leq \theta_j$ for $i \precsim j$ where $\precsim$ is a partial order on $\{1,2,\ldots,k\}$. Clearly, some parameter sets conform more closely to these order restrictions than others. We are interested in measures of the degree of conformity. Some of the measures available in the literature for the totally ordered case are generalized to the partially ordered case and the theory developed is applied to several tests of order restricted hypotheses.

AMS 1980 subject classifcation: Primary, 62F03; secondary, 62E15.

Key words and phrases: Order restricted inference, tests for and against a trend, isotonic inference, chi-bar-squared distributions, E-bar-squared distributions.

[1]This research was supported by the Office of Naval Research under the Contracts ONR N00014-80-C-0321 and ONR N00014-80-C-0322.

1. Introduction. In various situations, one is interested in a collection of parameters $\theta_1, \theta_2, \ldots, \theta_k$ which are believed to satisfy certain known order restrictions and inference procedures which make use of this ordering information are preferred. We consider order restrictions that are induced by partial orders on $\Omega = \{1,2,\ldots,k\}$. That is, suppose that $\preceq$ is a partial order on Ω and that the order restrictions are $\theta_i \leq \theta_j$ for all i and j with $i \preceq j$. Such a vector $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$ is said to be isotone (with respect to $\preceq$). In studying such inference procedures it is helpful to have a measure of the degree of conformity to the order restrictions. For instance, a test of H_0: θ is constant versus H_1: θ is isotone, but not constant should have power that increases with the degree of conformity. For a non-simple null hypothesis such a concept could be useful in identifying a least favorable configuration. In a Bayesian approach, priors which assign larger probabilities to parameters conforming more closely to the order restrictions would be sought.

Barlow, Bartholomew, Bremner and Brunk (1972) contains a thorough discussion of order restricted inference. Robertson and Wright (1982) develop several measures of conformity for the totally ordered case, ie. $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_k (1 \preceq 2 \preceq \ldots \preceq k)$. In considering unimodal structures, partial orders of the type $1 \preceq 2 \preceq \ldots \preceq r \succeq r+1 \succeq \ldots \succeq k$ arise and when making one-sided comparisons of several treatments with a common control, the partial order $1 \preceq i$ for $i = 2,3,\ldots,k$ occurs. (See Bartholomew (1959) and Robertson and Wright (1981).) Suppose that a dependent variable

has mean $\theta(i,j)$ when the first independent variable is fixed at level i, $1 \leq i \leq r$, and the second independent variable is fixed at level j, $1 \leq j \leq c$. If the levels are increasing and if $\theta(\cdot,\cdot)$ increases with each independent variable as the other is held fixed, then the order restrictions are $\theta(i,j) \leq \theta(s,t)$ for $i \leq s$ and $j \leq t$. This is another example of a partial order that is not total. We extend the measures of conformity in Robertson and Wright (1982) to the partially ordered case.

A set $L \subset \Omega$ is a lower layer provided $i \in L$ whenever $i \preceq j$ and $j \in L$. We denote the collection of lower layers by $\mathcal{L}$. To allow for different weights on the parameters, let $\underset{\sim}{w}$ be a positive weight function defined on Ω, ie. $\underset{\sim}{w} = (w_1,w_2,...,w_k)$. For situations in which the degree of conformity should be translation invariant, we consider the relationship $\gg$, defined on Euclidean space R^k, by $\underset{\sim}{x} = (x_1,x_2,...,x_k) \gg \underset{\sim}{y} = (y_1,y_2,...,y_k)$ if and only if

$$\sum_{i\in L} w_i(x_i - m(\underset{\sim}{x})) \leq \sum_{i\in L} w_i(y_i - m(\underset{\sim}{y})) \text{ for each } L \in \mathcal{L},$$

with $m(\underset{\sim}{x}) = \sum_{i=1}^k w_i\ x_i / \sum_{i=1}^k w_i$. Robertson and Wright (1982) argue that $\gg$ is appropriate for normal means, but for Poisson means a more appropriate measure is the following: $\underset{\sim}{x} \gg^* \underset{\sim}{y}$ if and only if

$$\sum_{i\in L} w_i\ x_i \leq \sum_{i\in L} w_i\ y_i \text{ for each } L\in\mathcal{L} \text{ and } \sum_{i=1}^k w_i\ x_i = \sum_{i=1}^k w_i\ y_i.$$

Remark 1.1. The relationship $\gg$ and $\gg^*$ are transitive and symmetric, $\gg^*$ is reflexive, and $\underset{\sim}{x} \ll \underset{\sim}{y}$ and $\underset{\sim}{x} \gg \underset{\sim}{y}$ imply that $\underset{\sim}{x}-\underset{\sim}{y}$ is a constant vector.

Proof. The first conclusion is obvious and because $\underset{\sim}{x} >> \underset{\sim}{y}$ is equivalent to $\underset{\sim}{x}-m(\underset{\sim}{x}) >>^* \underset{\sim}{y}-m(\underset{\sim}{y})$, it suffices to show that $>>^*$ is reflexive.

Suppose $\underset{\sim}{x} << {}^*\underset{\sim}{y}$ and $\underset{\sim}{x} >> {}^*\underset{\sim}{y}$. Let $L_0 = \phi$ and inductively define L_α to consist of those $j \in \Omega$ for which $i \preceq j$ and $i \neq j$ imply that $i \in L_{\alpha-1}$. Observe that $L_{\alpha-1} \subset L_\alpha$, $L_\alpha - L_{\alpha-1} \neq \phi$, and because Ω is finite, there is an integer h for which $\phi = L_0 \subset L_1 \subset \ldots \subset L_h = \Omega$. For each $j \in L_1$, $\{j\} \in L$ and so $x_j = y_j$. Next, for $j \in L_2$, $L(j) = \{i \in \Omega: i \preceq j\} \in L$, $L(j) - L_1 = \{j\}$ and so $x_j = y_j$. Continuing we see that $\underset{\sim}{x}=\underset{\sim}{y}$ and the proof is completed. $\square$

If one identifies vectors $\underset{\sim}{x}$ and $\underset{\sim}{y}$ which differ by a constant vector, then $>>$ induces a partial order on the equivalence classes which is essentially $>>^*$.

Let $C = \{\underset{\sim}{x} \in R^k: \underset{\sim}{x}$ is isotone with respect to $\preceq\}$ and note that the apriori belief concerning $\underset{\sim}{\theta}$ is that $\underset{\sim}{\theta} \in C$. Typically, estimates of $\underset{\sim}{\theta}$ are obtained by projecting initial estimates onto C, and test statistics are related to the distance from the initial estimates to the projections. The above measures of conformity can be characterized in terms of the Fenchel dual of C, which is defined by

$$C^{*\underset{\sim}{w}} = \{\underset{\sim}{y} \in R^k: \textstyle\sum_{i=1}^k w_i x_i y_i \leq 0 \text{ for all } \underset{\sim}{x} \in C\}.$$

(If $\underset{\sim}{w}$ is constant we denote the dual cone by C^*.) Barlow and Brunk (1972) and Dykstra (1981) discuss some of the implications of duality theory in order restricted inference. The following result is proved in the former reference (cf. Section 4).

Remark 1.2. With $\underset{\sim}{x},\underset{\sim}{y} \in R^k$, the following are equivalent:

(A) $\underset{\sim}{x} >> \underset{\sim}{y}$ $(\underset{\sim}{x} >>^* \underset{\sim}{y})$;

(B) $\underset{\sim}{y}-m(\underset{\sim}{y})-\underset{\sim}{x}+m(\underset{\sim}{x}) \in C^{*\underset{\sim}{w}}$ $(\underset{\sim}{y}-\underset{\sim}{x} \in C^{*\underset{\sim}{w}})$; and

(C) $\sum_{i=1}^k w_i(y_i-m(\underset{\sim}{y})-x_i+m(\underset{\sim}{x}))z_i \leq 0(\sum_{i=1}^k w_i(y_i-x_i)z_i \leq 0)$ for each $\underset{\sim}{z} \in C$.

Real valued functions which are nondecreasing with respect to these orderings are of interest. If f: $R^k \to R$ and $f(\underset{\sim}{x}) \geq f(\underset{\sim}{y})$ for all $\underset{\sim}{x},\underset{\sim}{y} \in R^k$ with $\underset{\sim}{x} >> \underset{\sim}{y}$ $(\underset{\sim}{x} >>^* \underset{\sim}{y})$, then f is said to be ISO(ISO*). The next result is immediate.

Remark 1.3. A function f: $R^k \to R$ is ISO if and only if it is ISO* and $f(\underset{\sim}{x} + c\underset{\sim}{e}_k) = f(\underset{\sim}{x})$ for all $\underset{\sim}{x} \in R^k$ and $c \in R$, where $\underset{\sim}{e}_k$ is a k-dimensional vector of ones.

Remark 1.4. Let $\underset{\sim}{x},\underset{\sim}{y} \in R^k$. $\underset{\sim}{x} >> \underset{\sim}{y}$ $(\underset{\sim}{x} >>^* \underset{\sim}{y})$ if and only if $f(\underset{\sim}{x}) \geq f(\underset{\sim}{y})$ for all f which are ISO(ISO*).

Proof. The result is an easy consequence of the definitions of ISO and ISO* and the following facts: $f_L(\underset{\sim}{x}) = -\sum_{i\in L} w_i(x_i-m(\underset{\sim}{x}))$ is ISO for each $L \in \mathcal{L}$, $g_L(\underset{\sim}{x}) = -\sum_{i\in L} w_i x_i$ is ISO* for each $L \in \mathcal{L}$ and $\sum_{i=1}^k w_i x_i$ is ISO*.$\square$

The partial ordering >>* is a cone ordering as discussed in Marshall, Walkup and Wets (1967) and the following result is contained in their work. However, its proof is so simple it is included here.

Theorem 1.5. Let f: $R^k \to R$ be differentiable and let $f_i(\underset{\sim}{x}) = \frac{\partial}{\partial x_i} f(\underset{\sim}{x})$ for $i = 1,2,\ldots,k$. If $f_i(\underset{\sim}{x})/w_i \leq f_j(\underset{\sim}{x})/w_j$ for all i and j with $i \preceq j$ and all $\underset{\sim}{x} \in R^k$, then f is ISO*.

Proof. Suppose $\underset{\sim}{x} >>^* \underset{\sim}{y}$. Using the mean value theorem there is a point $\underset{\sim}{z}$ on the line segment joining $\underset{\sim}{x}$ and $\underset{\sim}{y}$ for which

$$f(\underset{\sim}{y})-f(\underset{\sim}{x})=\sum_{i=1}^{k}(y_i-x_i)f_i(\underset{\sim}{z})=\sum_{i=1}^{k} w_i(y_i-x_i)(f_i(\underset{\sim}{z})/w_i)$$

and the latter sum is non-positive since $\underset{\sim}{y} - \underset{\sim}{x} \in C^{*\underset{\sim}{w}}$ and $(f_1(\underset{\sim}{z})/w_1,\ldots,f_k(\underset{\sim}{z})/w_k) \in C$ by hypothesis. $\square$

2. Preservation Theorems. In this section, we establish results which say that if $\underset{\sim}{X}$ is a set of observations, $f(\underset{\sim}{X})$ is a statistic with f ISO(ISO*) and $h(\underset{\sim}{\theta}) = E_{\underset{\sim}{\theta}}f(\underset{\sim}{X})$, then h is ISO(ISO*). The first result deals with a multinomial setting. Let $\underset{\sim}{w} = \underset{\sim}{e}_k$, let $A_n = \{\underset{\sim}{x} \in R^k$: each x_i is a nonnegative integer and $\sum_{i=1}^{k} x_i = n\}$, let $B = \{\underset{\sim}{p} \in R^k$: each $p_i \geq 0$ and $\sum_{i=1}^{k} p_i = 1\}$ and let $\underset{\sim}{X} = (X_1,X_2,\ldots,X_k)$ be a multinomial vector with parameters n and $\underset{\sim}{p} = (p_1,p_2,\ldots,p_k)$.

Theorem 2.1. If $f: A_n \to R$ is ISO, then $h(\underset{\sim}{p}) = Ef(\underset{\sim}{X})$ is ISO on B.

Proof. As in Robertson and Wright (1982), $h_i(\underset{\sim}{p}) - h_j(\underset{\sim}{p}) =$

$$\sum_{\underset{\sim}{y}\in A_{n-1}}(f(\underset{\sim}{y}+\underset{\sim}{\delta}_i) - f(\underset{\sim}{y}+\underset{\sim}{\delta}_j))n!\Pi_{i=1}^{k}(p_i^{y_i}/y_i!),$$

where $\underset{\sim}{\delta}_r$ is a k-dimensional vector with sth coordinate zero unless s = r and the rth coordinate is one. Suppose $i \precsim j$ and let $L \in \mathcal{L}$. If $i \notin L$ then $\sum_{r\in L}(\underset{\sim}{y}+\underset{\sim}{\delta}_i)_r = \sum_{r\in L}(\underset{\sim}{y}+\underset{\sim}{\delta}_j)_r$; if $i \in L$ and $j \notin L$, then $\sum_{r\in L}(\underset{\sim}{y}+\underset{\sim}{\delta}_i)_r \geq \sum_{r\in L}(\underset{\sim}{y}+\underset{\sim}{\delta}_j)_r$; and if $i,j \in L$, then $\sum_{r\in L}(\underset{\sim}{y}+\underset{\sim}{\delta}_i)_r = \sum_{r\in L}(\underset{\sim}{y}+\underset{\sim}{\delta}_j)_r$. The proof is completed by applying Theorem 1.5. $\square$

Chacko (1966) and Robertson (1978) considered testing H_0: $\underset{\sim}{p} = k^{-1}\underset{\sim}{e}_k$ with the alternative restricted by the trend, H_1: $\underset{\sim}{p}$ is isotone with respect to $\precsim$. Chacko considered the totally ordered case and Robertson the partially ordered case. The likelihood ratio test statistic is $T_{01} = -2 \ln \lambda = 2\sum_{i=1}^{k} X_i \ln(P(\underset{\sim}{X}|C)_i) - 2n \ln n + 2n \ln k$ where $P(\underset{\sim}{X}|C)$ is the

projection of $\underset{\sim}{X}$ onto C, which is characterized by

$$\sum_{i=1}^{k}(X_i - P(\underset{\sim}{X}|C)_i)P(\underset{\sim}{X}|C)_i = 0 \text{ and } \sum_{i=1}^{k}(X_i - P(\underset{\sim}{X}|C)_i)z_i \leq 0$$

for all $\underset{\sim}{z} \in C$. (See Barlow, Bartholomew, Bremner, and Brunk (1972, p. 28). Computation algorithms for $P(X|C)$ are also discussed in their Chapter 2.)

We first show that $f(x) = \sum_{i=1}^{k} x_i \ln(P(\underset{\sim}{x}|C)_i)$ is ISO on A_n, then note that this implies that $1_{[T_{01} \geq t]}$ is, for fixed t, ISO on A_n and applying Theorem 2.1, we see that the power function of T_{01}, $EI_{[T_{01} \geq t]}$, is ISO on B.

Suppose $\underset{\sim}{x} >> \underset{\sim}{y}$ with $\underset{\sim}{x},\underset{\sim}{y} \in A_n$, then $\underset{\sim}{y} - \underset{\sim}{x} \in C^*$ (we omit the superscript $\underset{\sim}{w}$ since it is constant) and so

$$\sum_{i=1}^{k} y_i \ln(P(\underset{\sim}{y}|C)_i) = \sum_{i=1}^{k} x_i \ln(P(\underset{\sim}{y}|C)_i) + \sum_{i=1}^{k}(y_i - x_i)\ln(P(\underset{\sim}{y}|C)_i).$$

The second term on the r.h.s. is nonpositive since $\underset{\sim}{y} - \underset{\sim}{x} \in C^*$ and $P(\underset{\sim}{y}|C) \in C$. Furthermore, $P(\underset{\sim}{x}|C)/n$ maximizes $\sum_{i=1}^{k} x_i \ln p_i$ with $\underset{\sim}{p} \in C$ and so $\sum_{i=1}^{k} x_i \ln(P(\underset{\sim}{y}|C)_i) \leq \sum_{i=1}^{k} x_i \ln(P(\underset{\sim}{x}|C)_i)$. Hence, $\sum_{i=1}^{k} y_i \ln(P(\underset{\sim}{y}|C)_i) \leq \sum_{i=1}^{k} x_i \ln(P(\underset{\sim}{x}|C)_i)$, or f is ISO on A_n.

The next result is an adaptation of Theorem 1.1 of Proschan and Sethuraman (1977). Let $\phi(\theta,x)$ be a nonnegative function defined on $(0,\infty) \times [0,\infty)$ satisfying the semigroup property,

$$\phi(\theta_1+\theta_2,x) = \int_0^\infty \phi(\theta_1,x-y)\phi(\theta_2,y)d\mu(y),$$

with μ either Lebesgue measure on $[0,\infty)$ or counting measure on the nonnegative integers.

<u>Theorem 2.2</u>. Let ϕ be as above, let $f: R^k \to R$ be ISO* and let h be defined on $(0,\infty)^k$ by

$$h(\underset{\sim}{\theta}) = \int_{[0,\infty)}\int_{[0,\infty)}\cdots\int_{[0,\infty)} f(\underset{\sim}{x}) \prod_{i=1}^{k} \phi(\theta_i,x_i)d\mu(x_1)\ldots d\mu(x_k),$$

where the integral is assumed finite. Then h is ISO*.

Lemma. For $i,j \in \Omega$, set $\delta_{ij} = \delta_i/w_i - \delta_j/w_j$. $C^{*\underset{\sim}{w}}$, the dual of the cone of isotone vectors, and K, the collection of vectors $\underset{\sim}{x} = \sum_{\{(i,j)\in\Omega^2:\ i\neq j\ i\preceq j\}} c_{ij}\delta_{ij}$ with the $c_{ij} \geq 0$ are equal.

Proof. A proof similar to that given for the Remark on p. 49 of Barlow, Bartholomew, Bremner and Brunk (1972) shows that

$$C^{*\underset{\sim}{w}} = \{\underset{\sim}{y}: \sum_{i\in L} w_i y_i \geq 0\ \forall L\in \mathcal{L} \text{ and } \sum_{i=1}^k w_i y_i = 0\}$$

For $L\in\mathcal{L}$, $\alpha,\beta \in \Omega$ with $\alpha \preceq \beta$ and $\alpha \neq \beta$,

$$\sum_{i\in L} (\delta_{\alpha,\beta})_i w_i = \begin{cases} 0 & \text{if } \alpha\notin L \\ 1 & \text{if } \alpha\in L \text{ but } \beta\notin L \\ 0 & \text{if } \alpha,\beta \in L. \end{cases}$$

So $K \subset C^{*\underset{\sim}{w}}$ and hence $K^{*\underset{\sim}{w}} \supset (C^{*\underset{\sim}{w}})^{*\underset{\sim}{w}}$. As Dykstra (1981) observed, $(C^{*\underset{\sim}{w}})^{*\underset{\sim}{w}} = C$ if C is a closed convex cone. This can also be shown using the following: the result holds when $\underset{\sim}{w} = \underset{\sim}{e}_k$, ie. $(C^*)^* = C$ for C closed, (cf. Rockafeller (1970, p. 121)) and $C^{*\underset{\sim}{w}} = \{(y_1/w_1,\ldots,y_k/w_k): \underset{\sim}{y} \in C^*\}$ (cf. Barlow and Brunk (1972)). Suppose that $\underset{\sim}{z} \in K^{*\underset{\sim}{w}} - C$, that is $\underset{\sim}{z}$ is not isotone and $\sum_{i=1}^k w_i z_i x_i \leq 0$ for each $\underset{\sim}{x} \in K$. Now if $\underset{\sim}{z}$ is not isotone there exist $\alpha,\beta \in \Omega$ with $\alpha \preceq \beta$, $\alpha \neq \beta$ and $z_\alpha > z_\beta$ and so $\sum_{i=1}^k w_i z_i (\delta_{\alpha,\beta})_i = z_\alpha - z_\beta > 0$. This contradiction implies that $K^{*\underset{\sim}{w}} = C$ or $C^{*\underset{\sim}{w}} = (K^{*\underset{\sim}{w}})^{*\underset{\sim}{w}} = K$. $\square$

Proof. (Theorem 2.2) Let $\underset{\sim}{w} = \underset{\sim}{e}_k$ and consider $\theta'' >>^* \theta'$, then $\underset{\sim}{\theta}' - \underset{\sim}{\theta}'' \in C^*$. Hence, $\underset{\sim}{\theta}' = \underset{\sim}{\theta}'' + \sum_{\{i\preceq j,\, i\neq j\}} c_{ij}\delta_{ij}$ with $c_{ij} \geq 0$. So it suffices to show that for arbitrary $\underset{\sim}{\theta}$, $h(\underset{\sim}{\theta} + c_{ij}\delta_{ij}) \leq h(\underset{\sim}{\theta})$, but this can be shown using the proof of Theorem 3.3 of Robertson and Wright (1982). $\square$

Suppose that k independent Poisson processes are each observed for T units of time and that the intensity of the ith process is θ_i. The likelihood ratio test of $\theta_1=\theta_2=\ldots=\theta_k$ when the alternative is restricted by the trend, $\underset{\sim}{\theta}$ is isotone, rejects for large values of

$$T_{01} = -2 \ln \lambda = 2\{\textstyle\sum_{i=1}^k X_i \ln (P(\underset{\sim}{X}|C)_i)-(\sum_{i=1}^k X_i)\ln(\sum_{i=1}^k X_i/k)\}$$

where λ is the likelihood ratio and $\underset{\sim}{X} = (X_1,X_2,\ldots,X_k)$ with the X_i independent Poisson variables and $E(X_i) = \theta_i T$. The family of Poisson densities satisfies the semigroup property with μ counting measure on $\{0,1,\ldots\}$, $-(\sum_{i=1}^k x_i)\ln(\sum_{i=1}^k x_i/k)$ is ISO* and we have seen earlier that $\sum_{i=1}^k x_i \ln(P(\underset{\sim}{x}|C)_i)$ is ISO*. Hence, Theorem 2.2 shows that this test has power function that is ISO*. This result could also have been obtained from Theorem 2.1 since conditioning on the total number of occurrences, $\sum_{i=1}^k X_i$, leads to a multinomial testing situation. However, this approach is more direct.

<u>Theorem 2.3</u>. Suppose $\{P_{\theta}: \underset{\sim}{\theta} \in \Theta\}$ is a family of probability measures on the Borel subsets of R^k with $\Theta \subset R^k$ and suppose that if $\underset{\sim}{X}$ has distribution P_{θ} then $\underset{\sim}{X}-\underset{\sim}{\theta}$ has distribution Q which is independent of $\underset{\sim}{\theta}$. If f: $R^k \to R$ is ISO and h: $\Theta \to R$ is defined by

$$h(\underset{\sim}{\theta}) = \int f(\underset{\sim}{x})dP_{\theta}(\underset{\sim}{x}) \quad \text{(which is assumed finite for each } \underset{\sim}{\theta} \in \Theta),$$

then h is ISO on Θ.

The proof of Theorem 2.3 is just like that given for the totally ordered case (cf. Robertson and Wright (1982)) and in fact, the result holds for any cone ordering (cf. Marshall, Walkup and Wets (1967)).

Suppose X_{ij}, $j = 1,2,\ldots,n$ and $i = 1,2,\ldots,k$, are independent normal variables with mean θ_i and common variance σ^2. The estimator $\hat{\sigma}^2 = \sum_{i=1}^{k}\sum_{j=1}^{n}(X_{ij}-\bar{X}_i)^2/(k(n-1))$ for σ^2 is independent of $\hat{\theta}_i = \bar{X}_i = \sum_{j=1}^{n}X_{ij}/n$. To test $\theta_1 = \theta_2 = \ldots = \theta_k$ with the alternative restricted by, θ is isotone, one could use $T = \sum_{\{i \preceq j, i \neq j\}}(\hat{\theta}_j - \hat{\theta}_i)/\hat{\sigma}$, or more generally

$$T_c = \sqrt{n}\ \sum_{i=1}^{k} c_i \hat{\theta}_i / [(\sum_{i=1}^{k} c_i^2)^{1/2}\hat{\sigma}] \text{ with } \sum_{i=1}^{k} c_i = 0.$$

Of course, this test rejects for $T_c \geq t$ where t is the $100(1-\alpha)$ percentile of the T distribution with $k(n-1)$ degrees of freedom. The power function is translation invariant, ie the power is the same at θ and $\theta + c e_k$, and so it is ISO if it is ISO*. The distribution of $\hat{\sigma}$ is independent of θ and the power at θ is given by

$$E(P_{\theta}[\sum_{i=1}^{k} c_i \hat{\theta}_i \geq t(\sum_{i=1}^{k} c_i^2)^{1/2}\hat{\sigma}/\sqrt{n} \mid \hat{\sigma}]).$$

So it suffices to show that for each positive a, $P_{\theta}[\sum_{i=1}^{k} c_i \hat{\theta}_i \geq at]$ is ISO*, but $\theta' \gg^* \theta$ implies that $\hat{\theta}' - \hat{\theta} \in C^*$ and so $\sum_{i=1}^{k} c_i(\hat{\theta}_i' - \hat{\theta}_i) \geq 0$ if $c \in C$. Hence if the vector c is isotone with respect to $\preceq$, then the power function is ISO.

In the case of T, c_i equals card. $\{\ell : \ell \preceq i\}$ - card. $\{\ell : \ell \succeq i\}$ which is easily seen to be isotone. For the simple tree ordering, $1 \preceq i$, $i = 2,\ldots,k$, this choice of c is $(-k+1,1,1,\ldots,1)$ and for the loop ordering, ie. $1 \preceq i \preceq k$ for $i = 2,\ldots,k-1$, this choice of c is $(-k+1,0,\ldots 0,k-1)$. The test for the simple tree case is discussed in Barlow, Bartholomew, Bremner and Brunk (1972 p. 188) and it is argued there that this choice of c provides the optimum set of scores.

Robertson and Wright (1982) consider the likelihood ratio test for this testing problem with a total order, unequal sample sizes and known variances which are not necessarily equal. The arguments given there also show that the likelihood ratio test in the partially ordered case has power that is ISO.

Robertson and Wegman (1978) developed the likelihood ratio test for H_1: $\underset{\sim}{\theta}$ is isotone with respect to $\preceq$ versus H_2: $\sim H_1$ for exponential families. In the normal means case with known variances and $w_i = n_i/\sigma_i^2$, the test statistic is $T_{12} = \sum_{i=1}^k w_i(\hat{\theta}_i - P_w(\hat{\theta}|C)_i)^2$ where $P_w(\cdot|C)$ denotes the projection with respect to the distance function $d^2(x,y) = \sum_{i=1}^k w_i(x_i-y_i)^2$. It is easy to show that neither T_{12} nor its negative is ISO*. As in Robertson and Wright (1982), we define another measure of conformity $x \gtrsim y$ provided $x - y \in C$. In the totally ordered case, $x \gtrsim y$ implies $x >> y$, but the converse is not true. However, in the partially ordered case this implication is not valid in general (For an example, consider k=3, the only order restriction is $2 \succeq 1$, $x = (0,0,0)$, $y = (1,1,-2)$ and $L = \{3\}$.) A function f: $R^k \to R$ is ISO** provided $f(x) \geq f(y)$ for all $x,y \in R^k$ with $x \gtrsim y$. The analogue of Remark 1.4, $x \gtrsim y$ if and only if $f(x) \geq f(y)$ for all f which are ISO**, is easy to establish. (Note that $f(x) = x_j - x_i$ is ISO** if $i \preceq j$.) Furthermore, since $\gtrsim$ is a cone ordering, Theorem 2.3 remains valid if ISO is changed to ISO**. Theorem 2.1 of Robertson and Wegman (1978) shows that the negative of $t_{12}(x) = \sum_{i=1}^k w_i(x_i - P_w(x|C)_i)^2$ is ISO**. So the modification of Theorem 2.3

which applies to $\gtrsim$ shows that if $\underset{\sim}{\theta} \gtrsim \underset{\sim}{\theta}'$, then the power of T_{12} at $\underset{\sim}{\theta}'$ is at least as large as $\underset{\sim}{\theta}$. Furthermore, $\underset{\sim}{\theta} \in C$ and $\underset{\sim}{\theta}'$ constant imply that $\underset{\sim}{\theta}-\underset{\sim}{\theta}' \in C$ or $\underset{\sim}{\theta} \gtrsim \underset{\sim}{\theta}'$. Hence, H_0: θ is constant is least favorable within H_1 and Robertson and Wegman (1978) have shown that under H_0, T_{12} has a chi-bar-squared distribution.

5. Comments. The problem of measuring the degree of conformity to an arbitrary partial order is a very broad one and in particular situations better measures may exist. In fact, we have noticed that none of the measures studied here are applicable in all the situations considered. In studying location parameters which are not related to the scale parameters, as in the normal case, $\gg$ is preferred, but for cases such as that of Poisson means, where the location and scale parameters are related, $\gg^*$ is more appropriate. We also found that $\gtrsim$ was useful when the null hypothesis stipulates that a collection of normal means satisfies a trend.

Because of the breadth of the problem it should not be surprising that in some special cases one can find a pair of parameter sets for which one of the orderings doesn't agree with our intuition. However, the measures studied here do seem to be useful in a variety of testing situations.

There are a couple of basic results in the totally ordered case which relate projections and the measures of conformity that are not true in the partially ordered case. Theorem 2.2 of Robertson and Wright (1982) states that

$$P_{\underset{\sim}{w}}(\underset{\sim}{y}|C) = \inf\{\underset{\sim}{z} \in C: \underset{\sim}{z} \gg^* \underset{\sim}{y}\}$$

and as a corollary $\underset{\sim}{x} \gg^* \underset{\sim}{y}$ implies $P_{\underset{\sim}{w}}(\underset{\sim}{x}|C) \gg^* P_{\underset{\sim}{w}}(\underset{\sim}{y}|C)$ and $\underset{\sim}{x} \gg \underset{\sim}{y}$ implies that $P_{\underset{\sim}{w}}(\underset{\sim}{x}|C) \gg P_{\underset{\sim}{w}}(\underset{\sim}{y}|C)$. The same example serves to show that these results are not valid in the general partially ordered case.

<u>Example</u>. Suppose that $k = 3$, $1 \prec 2 \succ 3$, $\underset{\sim}{w} = \underset{\sim}{e}_3$, $\underset{\sim}{x} = (0,4.5,4.5)$ and $\underset{\sim}{y} = (1,3,5)$. Observe that $\underset{\sim}{x} \gg^* \underset{\sim}{y}$ (and of course, $\underset{\sim}{x} \gg \underset{\sim}{y}$), $P_{\underset{\sim}{w}}(\underset{\sim}{x}|C) = \underset{\sim}{x}$, $P_{\underset{\sim}{w}}(\underset{\sim}{y}|C) = (1,4,4)$ (one could use the lower sets algorithm discussed in Barlow, Bartholomew, Bremner and Brunk (1972)), but $P_{\underset{\sim}{w}}(\underset{\sim}{x}|C) \gg P_{\underset{\sim}{w}}(\underset{\sim}{y}|C)$ is not true.

The Remark on p. 1236 of that paper is also not valid for arbitrary partially ordered situations. It states that if $\phi \neq A \subset R^k$ and A has a lower bound with respect to $\gg(\gg^*)$ then A has a greatest lower bound with respect to $\gg(\gg^*)$ and in the case of $\gg^*$ the greatest lower bound is unique. It is not difficult to construct examples with A a set with two elements which has a lower bound with respect to $\gg^*$ (and of course then with respect to $\gg$) but not a greatest lower bound.

REFERENCES

Barlow, R. E., Bartholomew, D. J., Bremner, J. M. and Brunk, H. D. (1972). Statistical Inferences Under Order Restrictions, Wiley, New York.

Barlow, R. E. and Brunk, H. D. (1972). The isotonic regression problem and its dual. J. Amer. Statist. Assoc. 67, 140-147.

Bartholomew, D. J. (1959). A test of homogeneity for ordered alternatives I, II. Biometrika 46, 36-48, 328-335.

Chacko, V. J. (1966), Modified chi-square tests for ordered alternatives. Sankhya B 28, 185-190.

Dykstra, Richard L. (1981). Dual convex cones of order restrictions with applications. Technical Report No. 111, Department of Statistics, University of Missouri-Columbia.

Marshall, A. W., Walkup, B. W. and Wets, R. J. B. (1967) Order-preserving functions: applications to majorization and order statistics. Pacific J. Math. 23, 569-584.

Proschan, F. and Sethuraman, J. (1977). Schur functions in statistics I. The preservation theorem. Ann. Statist. 5, 256-262.

Robertson, Tim (1978). Testing for and against an order restriction on multinomial parameters, J. Amer. Statist. Assoc. 73, 197-202.

Robertson, Tim and Wegman, Edward J. (1978). Likelihood ratio tests for order restrictions in exponential families, Ann. Statist. 6, 485-505.

Robertson, Tim and Wright, F. T. (1981). One-sided comparisons for treatments with a control. Technical Report No. 78, Department of Statistics, University of Iowa.

Robertson, Tim and Wright, F. T. (1982). On measuring the conformity of a parameter set to a trend with applications. Ann. Statist. 10, 1234-1245.

Rockafeller, R. T. (1970). Convex Analysis Princeton University Press, Princeton, New Jersey.

END

DATE
FILMED

9 — 83

DTIC